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Maximization as Creation Science

Leibniz argued that beauty is harmony, that harmony can be understood through number (that beauty—and being—is a product of number was the basis of Pythagorean cosmology) and that the mathematical principles of minimization and maximization (of how to maximize a dependent variable keeping an independent variable or independent variables to a minimum) were the fundamental principles of the mathematics of harmony (and, indeed, of cosmology and metaphysics). To understand Leibniz' ideas on minimization and maximization and their relation to harmony, they need to be given concrete significance. A wonderful book by the University of British Columbia mathematics professor Ivar Ekeland provides a context to Leibniz' ideas. In *The Best of All Possible Worlds: Mathematics and Destiny*, Ekeland sets out ideas that were debated around Leibniz' time and so allows us to understand the problematic to which Leibniz's theory of harmony was a response. He points out that the search for the simplest explanation of the cosmos was a traditional goal of physicists, astronomers, cosmologists, and philosophers (natural philosophers). When it came to motion of bodies (including heavenly bodies), it was assumed that circular motion was motion of the simplest sort. Most readers will recall that Plato describes the sphere as the most perfect of all figures in the *Timaeus* (33 b–c). Ptolemy (*Almagest* Book I.8) explains seemingly irrational motions of the planets by the combination of different circular motions. Lynne Ballew gives a quick synopsis of some the highlights of Greek tradition of according the circle and the sphere the status of representing what is perfect and harmonious. Hesiod was even more definite: he maintained that the completed universe is spherical in shape. Anaximander's map of the world indicated a circular surface. Empedocles' cosmos is a rounded sphere that enjoys the solitude of its self-containment. Pythagoras opined that contemplating the circular motion of the heavenly sphere imparts harmony to the soul of the contemplator. Alcmaeon's philosophy proposes that the motion of the soul imitates the circular motion of the stars and heavens. Anaxagoras' Nous causes the circular motion that orders the universe. Indeed, Ballew argues, with respect to both Plato and Parmenides, that

> (1) Being, which is "spherical," is apprehended by mind whose motion is circular. (For Parmenides, Being is stationary; "well-circled Truth: is its circumference, along which νοῦς, which thinks, truly proceeds. In the *Timaeus*, the universe as a whole rotates upon its axis, and the mind which thinks truly not only moves forward in a circular path but also revolves upon itself in imitation of the universal motion.)
> (2) Appearances, which shoot about in straight lines, are perceived by processes of opinion and sense perception which themselves consist of motion along straight paths.

Similar ideas dominated the perfection of circular motion dominated the medieval era, a fact that is hardly surprising, given that Calcidius' Latin translation and commentary on the *Timaeus* (ca. A.D. 360) was the only Platonic work widely known in the West prior to the systematic re-introduction of Greek philosophical works to the Latin West in the twelfth and thirteenth centuries. Indeed, even Galileo imagined circular motion to be the more nearly perfect than motion in a straight line.

Descartes's writings were revolutionary: Ekeland asserts that René Descartes "was the first one to state explicitly that linear and uniform motions are the simplest of all" (BPW 26). It was Descartes who first proposed the idea of what we now call "inertial motion," that is, the idea that a point travelling in space, free from any outside force, would move along a straight line at a constant speed. Descartes states in his *Principia philosophiae* (Principles of Philosophy, 1644)

> Altera lex naturae est: unamquamque partem materiae, seorsim spectatam, non tendere unquam ut secundum ullas lineas obliquas pergat moveri, sed tantummodo secundum rectas; etsi multae saepe cogantur deflectere propter occursum aliarum, atque, ut Paulo ante dictum est, in quolibet motu fiat quodammodo circulus, ex omni materia simul mota. Causa hujus regulae eadem est quae praecedentis, nempe immutabilitas et simplicitas operationis, per quam Deus motum in materia conservat.

Like nearly every advanced scientist of his time, Descartes had a deep interest in the science of optics. For Descartes, it is important to mention, science was not an empirical but a normative affair: reason dictates to nature the way it should act, and the task of scientist is, principally through pure reason, to discern those laws which reason gives to nature: what experience teaches, when it is ungrounded in reason, is of little value, for it does not inform us of nature's inner workings. In *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les science* (Discourse on Method Discourse on the Method of Rightly Conducting One's Reason and of Seeking Truth in the Sciences, 1637) Descartes announced

> Premièrement, j'ai taché de trouver en général les principes ou premières causes de tout ce qui est ou qui peut être dans le monde, sans rien considérer pour cet effet que Dieu seul qui l'a créé, ni les tirer d'ailleurs que de certaines semences de vérités qui sont naturellement en nos âmes. Après cela, j'ai examiné quels étoient les premiers et plus ordinaires effets qu'on pouvoit déduire de ces causes; et il me semble que par là j'ai trouvé des cieux, des astres, une terre, et même sur la terre de l'eau, de l'air, du feu, des minéraux, et quelques autres telles choses, qui sont les plus communes de toutes et les plus simples, et par conséquent les plus aisées à connoître.

One optical phenomenon which Descartes studied—he relied, of course, on deductive means—was refraction (the change in direction of light as it goes from one medium to another, for example, from air to water). *Le Dioptrique* compared the movement of light (which he believed is made up of tiny, tiny particles) to the trajectory of tennis ball (actually a ball used in *logue paume,* a popular game of Descartes time, which was played with rackets) as it crosses the boundary between media—Descartes had already discovered the means for resolving vectors into orthogonal components—while the vertical component is accelerated. This led Descartes to formulate what in commonly as Snell's Law but in France is referred to as "la loi de Descartes" or "loi de Snell-Descartes." The law states

Where

θ_1 is the angle between the the incident ray and the normal to the surface of the interface between the two media

Θ_2 is the angle between the refracted ray and normal to the surface of the interface between the two media

v_1 is the velocity of light in first medium

v_2 is the velocity of light in the second medium
n_1 is index of refraction of the first medium
n_2 is the index of refraction of the second medium

Descartes believed that velocity of light in air was actually less than the velocity of light in water. He explained his reasoning in *Le Monde ou Traité de la lumière* (a book Descartes completed in 1633 and sent off to be published, but withdrew when he he heard about the prosecution and sentencing of Galileo):

> Il me reste encore ici à vous faire considérer que l'action ou l'inclination à se mouvoir, qui est transmise d'un lieu en un autre par le moyen de plusieurs corps qui s'entre-touchent et qui se trouvent sans interruption en tous l'espace qui est entre deux, suit exactement la même voie par où cette même action pourroit faire mouvoir le premier de ces corps, si les autres n'étoient point en son chemin, sans qu'il y ait aucune autre difference sinon qu'il faudroit du temps à ce corps pour se mouvoir, au lieu que l'action qui est en lui peut, par l'entremise de ceux qui le touchent, s'étendre jusques à toutes sorts de distances en un instant.

Since the "several bodies" are more closely packed in water than in air, and so there is less space between them, these bodies transmit light more rapidly in water. So Descartes reasoned.

Pierre de Fermat (perhaps 1601, perhaps 1607 or 1608–1655), a lawyer who, by avocation, was a mathematician—indeed was a genius as a mathematician, disagreed with Descartes. In 1657 Fermat was sent a treatise, *On Light*, by Marin Cureau de la Chambre (1594–1669), which stated and derived the law of reflection (the angle of incidence of a lightray reflected from a surface is equal to the angle of reflection), which, like Descartes' was a deductive proof, though Cureau de la Chambre's proof was based on a different general principle: that nature will always take the shortest route for any action. The idea that nature acts thriftily and minimizes the expenditure of effort (however we understand that nebulous terms) has had a long history, reaching back to the first or second second of the common era (Hero of Alexandria, ca. 10 C.E.–70 C.E., already understanding that for the path of light reflecting from a mirror the angle of incidence equals the angle of reflection, showed that this path was the shortest length and least time) and even today it is still accepted today in the belief that objects in free fall follow the geodesic in mathematics, a geodesic (a generalization of the notion of a "straight line" to "curved spaces"—when a metric is available, geodesics are defined to be the shortest path between points on the space). This principle is often called the *lex parsimoniae*, and has a long and venerable history. Aristotle mentioned it, as did many others after him. Isocrates (436–338 B.C.E.) reportedly claimed that the small forces produce the motion of the large masses. Ptolemy (90–168 C.E.) based his model of the universe on the assumption that Ptolemy based his model on a notion that there is a divine urge toward symmetry. He realized that the circle is an efficient device for enclosing an area: the cirucumference of a circle encloses the greatest area in least possible perimeter. Moreover, a circle favours symmetry: equal arc lengths from the same circle will radially enclose equal areas. Malebranche proposed a metaphysical principle of the simplicity of ways. s'Gravesande, Leibniz, Wolff and others, until Maupertuis determined the law for the first time in a general formula. But it is Pierre Louis Moreau de Maupertuis (1698–1759; Maupertuis is the model for Voltaire's Dr.

Pangloss) whose name is most commonly associated with the “least action principle.”

> Après tant de grands hommes qui ont travaillé sur cette matiere, je n’ose presque dire que j’ai découvert le principe universel, sur lequel toutes ces loix sont fondées; qui s’étend egalement aux Corps durs & aux Corps élastiques; d’où dépend le Mouvement & le Repos de toutes les substances corporelles.
>
> C’est le pincipe de la *moindre quantité d’action*: principe si sage, si digne de l’Etre suprême, & auquel la Nature paroît si constamment attachée; qu’elle l’observe non seulement dans tous ses changemens, mais que dans sa permanence, elle tend encore à l’observer. *Dans le Choc des Corps, le Mouvement se distribue de manière que la quantité d'action, que suppose le changement arrivé, est la plus petite qu'il soit possible. Dans le Repos, les Corps qui se tiennent en équilibre, doivent être tellement situés, que s'il leur arrivoit quelque petit Mouvement, la quantité d'action seroit la moindre.*
>
> Les loix du Mouvement & du Repos déduites de ce principe, se trouvant precisement les mêmes qui sont observées dans la Nature: nous pouvons en admirer l’application dans tous les Phenomênes. Le mouvement des Animaux, la végétation des Plantes, la révolution des Astres, n’en sont que les suites: & le spectacle de l’Univers devient bien plus grand, bien plus beau, bien plus digne de son Auteur, lors qu’on sait qu’un petit nombre de loix, le plus sagement établies, suffisent à tous ces mouvemens. C’est alors qu’on peut avoir une juste idée de la puissance & de la sagesse de l’Etre suprême; & non pas lors qu’on en juge par quelque petite partie, dont nous ne connoissons ni la construction, ni l’usage, ni la connexion qu’elle a avec les autres. Quelle satisfaction pour l’esprit humain, en contemplant ces loix, qui sont le principe du Mouvement & du Repos de tous les Corps de l’Univers, d’y trouver la preuve de l'existence de Celui qui le gouverne!

Pierre de Fermat showed that the refraction of a beam of light that occurs as it passes from air is such that it follows the least action principle—or, rather, a version of the least action principle, though that version cast the principle in an even more remarkable mode than either Descartes or Marin Cureau de la Chambre had done. Fermat assumed, contrary to Descartes, that light travels more slowly in denser medium than it does in less dense medium (that it is propagated more slowly in water than in air). This allowed Fermat to arrive at a remarkable understanding about the path of the beam of light takes when it undergoes refraction: suppose I stand at the bank of a shallow river with a powerful light that emits a very, very narrow beam, and shine that light into the river and, since the river is shallow, I can see where where the light strikes the river bottom. I note where the spot on the water’s surface that the narrow beam illuminates, and I note where it reaches the bottom of the bottom of the river. I plot a line from the source of the illumination (the powerful light) to the spot on the water the light illuminates (let’s call this line AB) and then a line from that spot on the surface of the water to the point where it beam lands on the river bottom (let’s call this line BC). What Pierre de Fermat showed is that, given the difference in the speed of light in air and water, the time that it takes for the light to go from AB plus the time that takes for the light to go from BC is the requires the minimum amount of time for the light to go from A to C! Any other route from A to C would take the beam of light more time.

The least action principle raises quandaries. Suppose the university where I work were at the bottom of hill, and the house where I live at the top and that there were two routes to my house—one route involved a relatively climb up the hill, and then much shorter route across the top of the hill; the other route involved a long route (with no grade) across the bottom of the hill, then a relatively short path up the hill. Of course, I can figure out that since I am slowed down when I climb up a gradient, the second the route, with the shorter climb, will take less time than the first. I can understand how to minimize my effort and time because I can understand the temporal and corporal implications of lengthening and shortening the path traversed at a slower speed at the cost of greater effort—I can figure that out because I possess reason. But how does light know what path to take to minimize its time to get from AC—indeed how does it happen that out of the infinity of possible routes from A to C, the light will follow that path that minimizes the time required (that is, given the infinity of points D different than B, the time to travel from A to D plus the time to travel (at a different speed) from D to C will always be greater than the time it takes light to travel from A to B plus the time it takes for the light to travel (at a different speed) from B to C)? Leibniz himself used exactly this sort of calculation to explain the phenomenon of refraction: he considers the difficulty which light finds on passing through a medium, and he computes this difficulty by the path multiplied by the resistance. The ray always pursues that route in which the sum of the computed difficulties is the least; and according to this method *de maximis et minimis* he discovers the rule that minimizes "effort." In laying out this explanation, Leibniz has recourse to the idea of final causes which Descartes was so keen to reject. Newton in his *Optics* uses a version of the principle of the least resistance and in his *Principia Mathematica* Book 2, he determines what must be the meridian curve of a solid of revolution in order that the resistance experienced in that body in the direction of its axis may be the least possible.

Of course, we believe, nowadays, that what occurs in nature occurs without consciousness or purpose. How is it, then, then refraction acts to ensure the minimum expenditure of "effort"? It is not only people of recent times how sense the grip of the anomaly. In May 1662, Pierre de Fermat received two letters from the Cartesian, Claude Clerselier (1614–1684). In one, he wrote

> The principle upon which you build your proof, namely that nature always acts by the shortest and simplest ways, is but a moral principle, not a physical one, which is not and cannot be the c ause of any effect of nature. It is not, for it is not by this principle that it acts, but by the secret force and virtue which lies in every thing; the latter not being determined by that principle, but rather but the force that lies in all causes that concur to a single action, and by the dispositions which is found in all bodies on which the force acts. And it cannot be, otherwise we would be assuming some kind of awareness in nature; and by nature, we mean here only that order and that loaw which are established in the world as it is, and act without forethought, without choice, and by necessary determination (cited BPW 54–55).

Nature does not act to achieve purposes; there is no point in suggesting that nature strives to minimize the transit time of the ray of light, for that claim has no scientific validity. Nature does not pick one possibility from many with which it is confronted. There are not several doors from which it might choose—once the ray starts out, its path is determined.

Maupertuis differed. As we have seen, Maupertuis maintained that "in the collision of bodies, motion is distributed such that the quantity of action is as small as possible, given that the collision occurs. At equilibrium, the bodies are arranged such that, if they were to undergo a small movement, the quantity of action would be smallest." This principle is nothing less that a metaphysical principle, Maupertuis maintained, for it entails the existence of the One who governs the universe. Prince Rupert's Drops are a practical demonstration of the effects of nature's operating according to two interacting forces, minimization (the isoperimeter problem) and gravity: we noted above that glass bead that is a Prince Rupert's Drop assumes the shape of a water droplet with an elongated tail because it the surface tension of the molten glass tries to form a sphere—the form minimizes the energy acting on a surface to deform it—whilst being downward by gravity.

The prince of mathematicians, Leonhard Euler (1707–1783) concurred. In his grandly titled *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti* (A method for finding curved lines enjoying properties of maximum or minimum, or solution of isoperimetric problems in the broadest accepted sense, 1744) Euler wrote

> Cum enim Mundi universi fabrica sit perfectissima atque a Creatore sapientissimo absoluta, nihil omnino in mundo contingit, in quo non maximi minimive ratio quaepiam eluceat; quamobrem dubium prorsus est nullum, quin omnes Mundi effectus ex causis finalibus ope Methodi maximorum et minimorum aeque feliciter determinari queant, atque ex ipsiscausis efficientibus. ... Hoc modo curvatura funis seu catenae suspensae duplici via est eruta, altera a priori ex sollicitationibus gravitatis, altera vero per Methodum maximorum ac minimorum, quoniam funis eiusmodi curvaturam recipere debere intelligebatur, cuius centrum gravitatis infimum obtineret locum. Similiter curvatura radiorum per medium diaphanum variae densitatis transeuntium tam a priori est determinata, quam etiam ex hoc principio, quod tempore brevissimo ad datum locum pervenire debeant.

Euler claims, then, one can develop a mathematical physics grounded on the principle that God strives to minimize the amount of effort required to run the world: from this principle all the principles of physics can be derived. So the Prince of Mathematicians said.

Leibniz, earlier, was in substantial agreement.

> mais que les principes mêmes de la Mecanique ne scauroient estre expliqués Geometriquement, puis qu'ils dependent des principes plus sublimes, qui marquent la sagesse de l'auteur dans l'ordre et dans la perfection de l'ouvrage. Ce qui me paroist le plus beau dans cette consideration est que ce principe de la perfection au lieu de se borner seulement au general, descend aussi dans le partieulier des choses et des phenomenes, et qu'il en cst a peu pres comme dans la Methode de Formis Optimis, c'est à dir maximum aut minimum praestantibus, que nous avons introduite dans la Geometrie au delà de l' ancienne methode de maximis e t minimis quantitatibus. Car ce meilleur de ces formes ou figures ne s'y trouve pas seulement dans le tout, mais encor dans chaque partie; et meme il ne seroit pas d'assez dans le tout sans cela. Par exemple si dans la ligne de la plus courte descente entre deux points donnés, nous prenons deux autres points à discretion, la portion de cette ligne interceptée entre eux est encor necessairement la ligne de la plus courte descente à leur egard. C'est ainsi que les moindres parties de l'univers sont reglées suivant l'ordre de la plus grande perfection; autrement le tout ne le seroit pas.

In this same, rather late and very interesting text of 1694, “Tentamen Anagogicum” (*Philosophischen Schriften* hsg. Gerhardt Band VII, 270-279) Leibniz went on to relate this idea of minimization and maximization to Wilibrord Snellius’ formulation of the sine law of refraction and proposes his own explanation of that Snell’s law that is formulated in terms of minima and maxima. In another piece, written three years later, “Radical origination”, Leibniz writes (Gerhardt VII 305)

> Ita ergo habemus ultimam rationem realitatis tam essentiarum quam existentiarum in uno, quod utique Mundo ipso majus, superius anteriusque esse necesse est, cum per ipsum non tantum existentia, quae Mundus complectitur, sed et possibilia habeant realitatem. Id autem non nisi in uno fonte quaeri potest ob horum omnium connexionem inter se. Patet autem ah hoc fonte res existentes continue promanare ae produei produetasque esse, cum non appareat cur unus status Mundi magis quam alius, hesternus magis quam hodiernus ab ipso fluat. Patet etiam quomodo Dius non tantum physice, sed et libere agat, sitqutl in ipso rerum non tantum efficiens sed et finis, nec tantum ab ipso magnitudinis vel potentiae in machine universi jam constituta, sed et bonitatis vel sapientiae in constituenda, ratio habeatur.

Leibniz conceives creation as the choice among all possible worlds of the best world for actualization. The problem becomes one of maximizing goodness—that is, of maximizing richness and minimizing the complexity of laws so as to achieve a *harmonia rerum*. The boundary line separating the different worlds and forming the best possible world is, therefore, a minimax problem: the Principle of Perfection stipulates that certain qualities will be at a minimum or a maximum as the case may be. Descartes and Pierre de Fermat (and later Maupertuis) gave specific details about the operation of that principle.

An artwork would do well to imitate the Divine in this manner of operation, in this adherence to the Principle of Perfection. How does discern the mind of the Divine as it operates according to the Principle of Perfction? Through mathematics—that, I suggest, was Frampton’s gambit.

Or rather, it was, it it was not. The comments from Euler that I gave above were the last hurrah for the principle of least action (or the Principle of Perfection). After that a series of thinker, Lagrange, Jacobi, Hamilton, Mach all repudiated more and more assertatively the principle of final causes. Euler and Maupertius had proposed the principle of least action as an alternative to Newton’s mechanical conception of the universe: the great interest of Leibniz “Tentamen anagogicum” is that we see him arguing with considerable vehemence against the mechanical interpretation of principle of least action. It was only with Lagrange, Jacobi, Hamilton and Mach that the least action principle principle became part of the mechanical world view that has dominated Kuhnian “normal science” since Newton, as it was incorporated into the calculus of variations. From the time of Leibniz and Maupertius, to the time of Euler and then of Immanual Kant, the least action principle revealed a revealed a superior harmony or material teleology (a Zweckmäßigkeit) at work—a *Zweckmäßigkei*t whose operation was beyond the reach of empirical investigation. However, in Kant’s *Kritik der reiner Vernuft*, this idea of *Zweckmäßigkeit* became only a regulative idea, an regulative idea that later nineteenth and early twentieth century philosophers would repudiate altogether.

But it did come to dominate science, and the sort of explanation of the creator of world as seeking for harmony, which we can understand through number and mathematics, the world view of the Pythagoreans came to seem to hard-headed scientists as thoroughly implausible. Frampton, I suggest, was caught. I shall go on to argue that Frampton’s aesthetic is as aesthetic of harmony in the Renaissance neo-Pythagorean tradition. Or that was one side of his spiritual constitution—the other side had him a hard-headed set theorist. His way of integrating the two sides was treat the neo-Pythagorean aesthetic of harmony and cosmic order with a degree of irony, to distance himself from what he also accepted.

Minimization and the Knight's Tour

Of further relevance to Hollis Frampton's aesthetic theory is that the problem of the Knight's tour is also a problem of minimization—that is of least effort. For the question that common puzzle poses is whether a knight can be placed on the chess board and, following the usual rule concerning its its translation (it moves to the opposite end of 3 by 2 rectangle), be moved into every square on a chess board and, if so, in how many positions the Knight can be place dso that that tour of all the squares in the chess board can be accomplished. Another way of putting that question is, "given a grid of n rows and m columns representing a board, 6 ≤ n, m ≤ 10000, find the minimum number of starting positions you must examine, such that we can find all the solutions for the Knight's tour." For we know that certain symmetries of a rectangle mean that some routes are simply mirror images of others (and if one route lead to a solution, so will a given number of others).

For example, in the grid:

Maximization as the Clue to Harmony

But before we explore the Pythagorean implications of Leibniz' theodicy, we shall explore its implications for another topic dear to Hollis Frampton's heart, *viz.*, the Axiom of Choice. In this I am guided by a splendid article by Joel I. Friedman, "On Some Relations between Leibniz's Monadology and Transfinite Set Theory: A Complement to Russell's Thesis on Leibniz." Bertrand Russell had argued that Leibniz' logic is the key to his metaphysical theories; Friedman argues the complementary view that Leibniz' metaphysics serves as an intuitive guide to transfinite set theory. Specifically, Friedman sets out to show that certain maximizing processes in Leibniz' monadology are analogous to maximizing processes in transfinite set theory. Friedman explains that "a set-theoretical maximising principle is roughly any statement which maximises the number of sets of a certain kind, or the number of sets in a set a certain kind." Leibnizian maximizing principles, on the other hand, are consequences of the Principle of Perfection, a principle God follows in deciding which possible world to create. The basic analogy that Friedman lays out is that just as God follows certain maximizing principles in deciding which possible world to make actual, so in set theory corresponding maximizing principles decide set theoretical statements.

To understand this analogy, we have to recall Leibniz' theory of monads. Leibniz' theory of monad was developed in opposition to the Cartesian conception of physical substance. In Meditation V of (Meditationes de prima philosophia, in qua Dei existentia et animæ immortalitas demonstratur (Meditations on First Philosophy), Descartes argued that the essence of material substance is extension, the property of filling up space—his geometry was a method for studying extension, and the possibility of dividing a uniform space into distinct parts. Descartes inferred from this that in reality only a single extended substance exists, a single substance which comprises all matter. Individual bodies are merely modes of the one extended substance; he inferred, too, that there can be no void (since there can be no space without extensions, i.e., substance) and, in a work entitled *Le Monde*, set out his conviction that all motion has the nature of a circular vortex. Further, in Meditation VI he argued that since bodies are essentially extension, and extension is understood through number and ratiocination, the true nature of bodies is apprehended by pure thought; information from the senses contributes nothing essential.

Against Descartes, Leibniz maintained that for any being to be a being, it must be simple. Extension, however, is divisible. This led him to conclude that extension itself cannot constitute substance, that substance must have some metaphysical support. He sometimes described this metaphysical support as 'formal atoms,' as opposed to material atoms (such as had been proposed by Newton and Locke, and the conception of which he deemed to contain a contradiction, since what is material can be further divided, and so cannot be atomic). These monads, along with God, are all that is real. Monads are non-composite, immaterial, soul-like. A monad is the reality of a complete concept, i.e., a concept that contains within itself all the predicates of the subject of which it is the concept. As a complete concept, it contains, folded up within itself, all its past properties—at least, it must bear the trace of all past properties—and contains virtually all the properties it will exhibit in the future, that will be unfolded through time, as sufficient reasons for their appearing arise. This leads Leibniz to a theory of time that distinguishes between three orders of time:

a) the reality: God's eternal, atemporal being.

b) the continuous, and immanent, self-becoming of monads (the monad's entelechic self-becoming)

c) the illusory time of an chronology of "nows" external to the monads.

These unusual ideas, of reality as essentially mental or spiritual and as harbouring the past and future in itself, and of time as illusion are consonant with ideas we seen Hollis Frampton propose. (Thus, Leibniz asserts, "Les relations et les ordres ont quelque chose de l'estre de raison, quoyqu'ils ayent leur fondement dans les choses; car on peut dire que leur realité, comme celles des verités eternelles et des possibilités vient de la supreme raison". Or, again, "Les Relations ont une realité dependante de l'Esprit comme les Verités; mais non pas de l'Espirit des hommes, puisqu'il y a une supreme intelligence qui les determines toutes de tout temps.") But the more remarkable consonance is with the form of reason he employs to work out these ideas, one associated with set theory.
Friedman proposes that

> Leibniz's monadology has a set-theoretical character. The universe may be regarded as a set of monads with the structure imposed on the set. Every physical body may be regarded as a set of monads with a structure imposed on the set. And, finally, every monad itself may be regarded as (or at least correlated with) a set or sequence of perceptions with a structure imposed on the sequence. Moreover, vertically and horizontally throughout, there are properties of and relations among the perceptions, the sequence of perceptions, the structured sets of these sequences, and the whole system of these structured sets. Thus, the monadology presents a very complicated set-theoretical character, indeed, a transfinite set-theoretical character. [cf. "elle va même au delà des combinaisons finies, elle en fait une infinité d'infinies, c'est à dire une infinité de suites possibles de l'univers, don't chacune contient une infinité de creatures."]

We could formulate an analogy (though we must be vigilant and not allow this analogy to reduce Friedmann's argument).

> The film may be regarded as set of images [frames] with a structure imposed on the set. Every set of images attached to a letter designator (the set of images associated with'a,' both the word-pictures of 'a'-words and the images of someone turning pages of a book; the set of images associated with 'b,' that is,

> word-pictures of ‘b’-words and the images of eggs frying in a skillet; the set of images associated with ‘c,’ both the word-pictures of ‘c’-words and the image of the red ibis flapping its wings) with a structure imposed on the set. And finally, every frame itself may be regarded as (or correlated with) a set or sequence of frames (shots) with a structure imposed on the sequence. Moreover, vertically and horizontally throughout, there are properties and relations among shots, the structured sets of these sequences, and the whole system of structured sets. Thus *Zorns Lemma* presents a very complicated set-theoretical character, indeed, a transfinite set-theoretical character.

Friedman continues by recasting into somewhat formal terms (though, his approach is still quite intuitive) that principle of Leibniz we recorded above, that God choses that universe that at the same time is the simplest in hypotheses and the richest in phenomenon. (“Mais Dieu a choisi celuy qui est le plus parfait, c’est à dire celuy qui est en même temps le plus simple en hypotheses et le plus riches en phenomenes comme pourroit estre une ligne de Geometrie don’t le construction seroit aisée et les proprietés et effets seroient fort admirables et d’une grande étendue.”) Here is Friedman’s reworking of that principle.’

> [W]e may state for Leibniz that the most perfect combination of monads is the greatest combination of compossible monads, with the greatest possible variety and the greatest possible order (thus simplest in its laws). And thus we may state the following maximizing principle . . .
>
> $\boldsymbol{L_1}$ — The universe contains the greatest combination of compossible monads, with the greatest possible variety and the greatest possible order (thus simplest in its laws).
>
> It is important to note that, for Leibniz, the notion of ‘most perfect’ or ‘best’ reduces to a maximising (or optimising) notion. And so God decides what should exist by maximising existence (subject to contraints). We certainly agree with Rescher’s statement, “Thus the Principle of Perfection is a maximum principle, and it furnishes the mechanism of God’s decision among the infinite mutually exclusive systems of compossibles.”

Friedman goes on to develop a principle in set theory analogous to $\boldsymbol{L_1}$. To develop that principle, Friedman has to draw on the distinction between classes and sets and the notion of proper classes. A proper class is defined a class which is not the element of any class. A set is defined as a class which is an element of some class. Thus a class is either proper class or set (but not both). The sorts of paradoxes that Russell identified early in the twentieth century are avoided by stipulating that proper classes, since they are not elements of any classes, cannot be operated on in the same way that sets can be.
Friedman continues

> It was von Neumann who first formalised the distinction between proper classes and sets. Moreover, he was bold enough to conjecture that the universal class ***V*** [consider this as analogous to Leibniz’ universe of monads] is maximised in the sense that ***V*** contains as an element every class which is not in one-one correspondence with ***V***.

Mathematicians say that a class, ***A***, is one-one correspondence with another class, ***B***, when one can match every element in ***A*** with an element in ***B***, with no elements in either ***A*** or ***B*** left over. This allows Friedman to state thus von Neumann's Maximizing Principle, which he calls ***MP***

V is maximised when $((\forall X)\ (X \not\approx V \rightarrow X \in V)$

That is, the Universal Class (again, consider, this as an analogue of Leibniz' universe of monads) is maximized when, for any class ***X***, if class ***X*** is not in one-one correspondence to class ***V*** (that is, when you cannot match up every element in ***V*** with an element in ***X***, with no elements in a either ***V*** or ***X*** left over) then class ***X*** belongs to ***V***. Friedman notes that "***MP*** is a maximising principle because, intuitively speaking, ***MP*** implies that ***V*** contains just about as many sets as possible, for any class which ***V*** does not contain is in one-one correspondence with ***V*** and thus 'too big' to be an element of ***V***. Indeed, ***MP*** is equivalent to the statement that every proper class is in one-one correspondence with ***V***."
Friedman points out that ***MP*** is analogous to Leibniz' principle L_1 for just as L_1 asserts that every universe contains the greatest number of compossible monads, with the greatest possible variety and with the greatest possible order, so ***MP*** implies that the set-theoretical universe has the maximum number of sets, with a great amount of variety and a great amount of order. Of course, any attentive reader would note, and want to challenge, the slippage from 'greatest' to 'great' in those last two phrases, but dealing with that fascinating topic would divert us from our business, which is to indicate that set theory, and maximization principles, might be taken as providing analogies to aesthetic principles, and even to onto-aesthetic principles (and that, indeed, I am convinced is how Hollis Frampton took them). Our business, then, is to make concrete the metaphysical and onto-aesthetic principles that follow from the maximization operations that Leibniz saw God as performing and Frampton (as we have seen) saw the artist as performing.
For this, we return to Friedman. Friedman points out that

> according to Leibniz, every monad, however confusedly, represents the entire universe from its point of view [as Frampton represented "a universe" from his point of view in *Travelling Matte*, a point of view as isolated as that of Leibniz's monads, which do not directly communicate with (in the sense of having direct effects on) one another]. Moreover, it represents it more clearly from anear than from afar.
> Again, let us consider various passages. Consider for example what Leibniz says in section 56 of the *Monadology*.
>
>> Now this interconnection, relationship or adaptation of all things to each particular one, and of each one to all the rest, brings it about that every simple substance has relations which express all the others and that it is consequently a percpetual living mirror of the universe.

Also consider article IX of the *Discourse on Metaphysics*.

>> That every individual substance expresses the whole universe in its own manner and that in its full concept is included all its experiences together with all the attendant circumstances and the whole sequence of exterior events.

> And within article IX, consider.

> Furthermore every substance is like an entire world and like a mirror of God, or indeed of the whole world which it portrays, each one in its own fashion; . . . . It can indeed be said that every substance bears in some sort the character of God's infinite wisdom and omnipotence, and imitates him as much as it is able to; for it expresses, although confusedly, all that happens in the universe, past, present and future, deriving thus a certain resemblance to an infinite perception or power of knowing. And since all other substances express this particular substance and accommodate themselves to it, we can say that it exerts its power upon all the others in imitation of the omnipotence of the creator.

These remarks lead Friedman to the conclusion that each monad has as many perceptions as are possible from its vantage point. Morever, though the perceptions of monads are so less perfect than God's, each monad's perceptions are as perfect as they can be from its point of view. God is the ultimate monad, and His perceptions represent perfectly all other monads, while other monads represent, imperfectly and from its own vantage point, all other monads. Friedman concludes this section by endorsing Mates principle

> possible worlds are maximal sets of mutually compossible complete individual concepts [monads] and a complete individual concept [monad] is a maximal set of (or a "maximal" attribute composed of) compatible simple attributes [perceptions]. [It is left to reader to ponder the analogies to an aesthetic that states that every particle (every element of the most basic sort) in an artwork combines maximum of variety with sheer simplicity.]

This leads Friedman to the conclusion that, by condensing the passages above, his maximizing principle for sets can be rephrased:

> $\boldsymbol{L_2}$ — Every monad represents the entire universe from its point of view with the greatest number of perceptions it can have. [It is left to the reader to ponder the analogies to an aesthetic principle that states that every element in an artwork must possess a maximum of variety.]

Expanding on this, Friedman says that the greatest perfection of the universe is obtained if the perceptions for each monad are as perfect as possible. Both $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$ are maximizing principles. God acts in accordance with both principles to decide which possible worlds, and which possible monads, to create.

$\boldsymbol{L_1}$ was a maximization principle for the universe, $\boldsymbol{L_2}$ a maximization principle for monads with the universe. We have, on analogy with $\boldsymbol{L_1}$, a maximization principle ***MP*** that applies to the universal class. We need, as an analogy for $\boldsymbol{L_2}$, a generalized maximization principle that applies not to the universal class, but to classes within the universal class. In so far as maximizationn concerns the universal class ***V***, we can say a class (in this case V) is maximized if it contains as an element every class that is not equinumerous with V.

We can generalize that notion of the maximization of a class to say that a class is maximized if it contains as an element every subclass of that class which is not equinumerous with that class. This is our ***GMP***: every class is maximized, in the sense of 'maximized' just given. Friedman points out the ***GMP*** is analogous to L_2, for, just as L_2 asserts that every monad may be taken as a maximal set of perceptions and that every monad is in a hierarchy with other, similar maximal sets, so ***GMP*** asserts that every local universe contains the maximum number of sets and is in a hierarchy with other local universes. L_1 and L_2 maximize the whole and each of its parts; analogously ***MP*** and ***GMP*** maximize the whole and arbitrarily many of its parts. This leads Friedman to the conclusion that L_1 and L_2 decide important metaphysical issues, while ***MP*** and ***GMP*** analogously decide important set-theoretical issues for transfinite sets.

This has been an extended foray into some key ideas relating to principles of maximization of sets. What relevance has this to Frampton's artistic principles and practices? First, the point of his work, I have claimed, is to suggest how consciousness grasps reality—how it acquires metaphysical or ontological insight. Frampton makes an extraordinary—and what can seem quite unwarranted conjecture: the set theory models consciousness, including consciousness efforts at acquiring metaphysical or ontological insight.

Frampton's conjecture might appear to be far-fetched, but, guided by Friedman, our foray into Leibnizian metaphysics and its set-theoretical analogues has shown 1) that Frampton had impressive predecessors for thinking as he did 2) that understanding the analogy between set theory and mathematics affords insights into the rational order of being, another seemingly extravagant belief that, on the explicit evidence of his writings as well of the implied evidence of the forms he forged for his film, Frampton refused to abandon (as is the fashion of the times). This last point leads us to venture to extend the point of the examination. For Frampton, rational order, metaphysical order and aesthetic order are one: all concern the ideal of Being as the Beautiful. Concerning this proposal, recall that Frampton did suggest that he understood artmaking as choosing to actualize one possible configuration of items out of the set of all possible configurations of a given set of items. We could point out that the film provides for even more exact comparisons between the items that make up the actualized film and Leibniz' idea of monads. Extending this by introducing the idea of aesthetic value, we could say that the artist chooses to actualize the best configuration of items from amongst all possible configurations. (We could easily weaken this proposed extension of the relevance of set theory to aesthetics by suggesting that because the artist possesses a less than perfect understanding of the items that compose the work, so he or she cannot hope to actualize the best order, but only a good configuration.) Thus, the form of *Zorns Lemma* provide insight into the nature of art making. It would be fruitful to examine the work to discern precisely what the form of the work has to say about the nature of aesthetic order, about the coexistence of different orders of order, about mirroring of the whole in the part, about the harmony of the parts, etc. Considering these questions in Leibnizian terms, or, even better, in terms of Leibniz and Malebranche's debate over theodicy, would be most enlightening. Unfortunately, I have not the space to do so here. But, as do authors of mathematical texts that leave problems for the reader, I make a few suggestions

Recall that Leibniz claimed God choses that universe that at the same time is the simplest in hypotheses and the richest in phenomenon. ("Mais Dieu a choisi celuy qui est le plus parfait, c'est à dire celuy qui est en même temps le plus simple en hypotheses et le plus riches en phenomenes comme pourroit estre une ligne de Geometrie don't le construction seroit aisée et les proprietés et effets seroient fort admirables et d'une grande étendue.") In the somewhat polemical "Von der Weisheit" (On Wisdom), Leibniz asserts that "Glückseligkeit, Lust, Liebe, Vollkommenheit, Wesen, Kraft, Freiheit, Übereistimmung, Ordnung und Schönheit aneinander verbunden, welches von Wenigen recht angesehen wird." He explained.

> alles Wesen in einer gewissen Kraft bestehet, und je größer die Kraft, je höher und freier ist das Wesen.

> Ferner bei aller Kraft, je größer sie ist, je mehr zeiget sich dabei Viel aus einem und in einem, indem Eines viele außer sich regieret, und in sich vorbildet. Nun die Einigkeit in der Vielheit ist nichts anders, als die Übereinstimmung, und weil eines zu diesem näher stimmet, als zu jenem, so fließet daraus die Ordnung, von welcher alle Schönheit herkommt, und die Schönheit erwecket Leibe.
> Daraus siehet man nun, wie Glückseligkeit, Lust, Liebe, Vollkommenheit, Wesen, Kraft, Freiheit, Übereistimmung, Ordnung und Schönheit aneinander verbunden, welches von Wenigen recht angesehen wird.

Let's examine this a little more carefully, for there are subtleties in it. Take the phrase "so fließet daraus die Ordnung, von welcher alle Schönheit herkommt" (thus the order, from which all beauty comes forth, flows from it). First, a remark concerning the phrase "the order from which beauty comes forth" —I believe we can take the phrase as being restrictive: that is, I believe we can take the phrase "the order from being being comes forth" to imply there is a type of order from which beauty comes forth, and another type (or types) of order from which no beauty arises. This interpretation is consistent with ideas that Leibniz propounds in *Discours de metaphysique*. Let us look more thoroughly at the text we considered above, concerning attempting to draw a line through points that have been put on paper helter-skelter.

> Je dis qu'il est possible de trouver une ligne géométrique dont la notion soit constante et uniforme suivant une certaine règle, en sorte que cette ligne passe par tous ces points, et dans le même ordre que la main les avait marqués. Et si quelqu'un traçait tout d'une suite une ligne qui serait tantôt droite, tantôt cercle, tantôt d'une autre nature, il est possible de trouver une notion, ou règle, ou équation commune à tous les points de cette ligne, en vertu de laquelle ces mêmes changements doivent arriver. Et il n'y a, par exemple, point de visage dont le contour ne fasse partie d'une ligne géométrique et ne puisse être tracé tout d'un trait par un certain mouvement réglé. Mais quand une règle est fort composée, ce qui lui est conforme passe pour irrégulier.

There is no collection of points, however helter-skelter they are, for which no line passing through (or near) these points can be found, and there is no line, however many changes in direction it undergoes, that cannot be represented by a rule. However, the more complex this rule or formula becomes, the more we deem the array of points to be irregular. Any universe that God might have created would have been regular and order for "Dieu ne fait rien hors de l'ordre et il n'est pas même possible de feindre des événements qui ne soient point réguliers." However, Leibniz asserts that God chooses the universe that is at the same time the simplest in hypotheses and the richest in phenomena, "Dieu a choisi celui qui est le plus parfait, c'est-à-dire celui qui est en même temps le plus simple en hypothèses et le plus riche en phénomènes, comme pourrait être une ligne de géométrie dont la construction serait aisée et les propriétés et effets seraient fort admirables et d'une grande étendue."
I want to make several points regarding this assertion. First, the lines represents a simple form that integrates a wide variety of features and its principle (the formula that can be used to represent the line) has a many implications (what it represents is "le plus riche en phénomènes"). Thus, in Leibniz' terminology, the line possesses harmony: "Nun die Einigkeit in der Vielheit ist nichts anders, als die Übereinstimmung" (Now, the Oneness in the Many is nothing other than harmony). In a letter to Christian Wolff of May 18, 1715, Leibniz identifies perfection (and recall that "le plus parfait is "celui qui est en même temps le plus simple en hypothèses et le plus riche en phénomènes") with "harmonia rerum" (the harmony of things), or the "observabilitas universalium" (the "observability of universals"), or, "consensus vel identitas

in varietate" ("concord or identity in variety"). For Leibniz, a possible world's harmony varies with its perfection, and so if God chooses to actualize that world which is the most perfect, then, in actualizing that world, He will actualize the world which is the most harmonious. And whatever the differences between the Pythagorean and the Leibnizian conception of harmony (and they are many), the two conceptions do overlap; and the idea that form of the universe is governed by harmony is principle to which both Pythagoras and Leibniz would have given consent. Thus, we see that our foray into the Leibnizian aesthetic allows us (helped by Friedman) to make an extraordinary connection between the axiom of choice (and maximization principles) and harmony.

Perfection—and therefore harmony—produce pleasure: "Je crois que dans le fonds le plaisir est un sentiment de perfection et la douleur un sentiment d'imperfection" (I believe that basically pleasure is feeling of perfection and pain a feeling of imperfection) In Leibniz was more expansive in "Von der Weisheit."

> Wenn nun die Seele in ihr selbst eine große Zusammenstimmung, Ordnung, Freiheit, Kraft, oder Vollkommenheit fühlet, und folglich davon Lust empfindet, so verursachet solches eine Freude, wie aus allen diesen und obigen Erklärungen abzunehmen.
>
> Solche Freude ist beständig und kann nicht betrügen, noch eine künftige Traurigkeit verursachen, wenn sie von Erkenntniß herrühret, und mit einem Licht begleitet, daraus im Willen eine Neigung zum Guten, das ist die Tugend, entstehet.
>
> Wenn aber die Lust und Freude so bewandt, daß sie zwar die Sinnen, doch aber nicht den Verstand vergnüget, so kann die ebenso leicht zur Unglückseligkeit, als zu Glückseligkeit helfen, gleichwie eine wohlschmeckende Speise ungesund sein kann.
>
> Und muß also die Wollust der Sinnen nach den Regeln der Vernuft, wie eine Speise, Arznei oder Stäkung gebraucht werden. Aber die Lust , so die Seele an sich selbst, nach dem Verstand, empfindet, ist eine solche gegenwärtige Freude, die uns auch vors Künftige bei Freude erhalten kann.

Leibniz understood well the connection of being and beauty in his work; the analogy between the Creator and the artist who, in Frampton's terms, selects from many possible ways of organizing his or her material, one particular (and hopefully satisfying) possibility. Harmony is a key attribute of the universe. Leibniz asserted that "il est de la sagesse de Dieu, que tout soit harmonique dans ses ouvrages" and that God has instantiated "la plus parfait des Harmonies." Leibniz drew from his ideas about permutational form, the idea that "Toutes les pensées ne soit rien d'autre que des simples complications des idées, comme les mots de lettres d'alphabet." (A VI iii 413) This is a remarkable consonant the 'modernist' idea of composition that Hugh Kenner set out in "Art in a Closed Field" and *Flaubert, Joyce, Beckett: The Stoic Comedians*; we have seen that those works exercise a considerable influence of the young Hollis Frampton.

There is more to say on the connection of these Leibnizian ideas to *Zorns Lemma*. However, to deal with those relations requires us to understand Leibniz' notion of "universal harmony." Leibniz characterizes his idea 'universal harmony' as the realm of grace within the realm of nature. Universal harmony is a sort of sympathy amongst all monads, so that what happens in any one of theme will be mirrored in all others (with an intensity that is inversely proportional to their distance one from the source). Lloyd H. Strickland offers an intriguing possible interpretation of Leibniz' conception of harmony:

> perhaps the harmony of things in the Leibnizian best possible world is nothing

> more than God's own essence diversely manifested in the creatures of this world. This characterization squarely satisfies the requirement for 'unity in variety,' for . . . Leibniz adopted the Neo-Platonic view that in creating populating the world, God diffuses a finite portion of his essence to all created things. Thus every created thing possesses a certain degree of this 'God stuff,' and nothing else besides. . . . If the essence of God, diversely manifested, is ultimately all there really is in the world then this obviously entitles Leibniz to identify a unity, or basic sameness in all things. And if, as Leibniz maintained, no two creatures possess the same essence, i.e., the same degree of perfection, then the world will exemplify the variety that, together with its underlying unity, gives rise to harmony..

Christina Mercer makes the same point in *Leibniz's Metaphysics*. She points out that universal harmony has two constituents.

> The first of these, what I call *Emanative Harmony*, follows from the assumption that God creates and maintains the world through emanation, and therefore that every creature is an instantiation of the divine essence. The second of these, what I call *"Reflective Harmony*," is a version of the Platonist Theory of Reflective Harmony.
> Leibniz's original conception of harmony is a combination of Emanative and Reflective Harmony, where the Supreme Being emanates its essence to creatures, some of which are in Reflective Harmony with one another.

Leibniz viewed grace essentially as the presence of the Divine in things. This when he wrote "il est de la sagesse de Dieu, que tout soit harmonique dans ses ouvrages," he continued, "et que nature soit parallele à la grace." The following passage, Leibniz comes very close to stating idea we are developing here, *viz*., that the Leibnizian notion of harmony identifies it with God's presence in things.

> "The conceivables themselves must contain the reason why they are sensed, that is why they exist. But the reason is not (contained) in the thought of single (things). It must, therefore, be in (the thought) of a plurality. Therefore, in that of all (things). Therefore, in the Mind, which is one in many. Therefore, in Harmony, i.e., the unity of many, or diversity compensated by identity. God, however is one in all." ("Elements of mind and body "A VI ii 276–291, here 283)

Leibniz's views on emanation parallel reasonably closely those that Robert Grosseteste expounded, in "De Luce (De Inchoacione Formarum)" and Grosseteste, of course, also described this emanation in terms of harmony. Moreover, as both Mercer and Strickland point out, Leibniz' theory of emanation was essentially a hylomorphic theory, as was Grossesteste's theory of emanation. This connection suggest the depth of the relations amongst the topics of harmony, light, emanation, set theory, possible worlds, beauty, unity, multiplicty, difference, edges, and love that Frampton has drawn together.